



On the convergence of finite difference methods for weakly regular singular boundary value problems

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Abstract

The second order finite difference methods M_1 based on a non-uniform mesh and M_2 based on an uniform mesh developed by Chawla and Katti [Finite difference methods and their convergence for a class of singular two point boundary value problems, Numer. Math. 39 (1982) 341–350] for weakly regular singular boundary value problems $(p(x)y')' = f(x, y)$, $0 < x \leq 1$, with $p(x) = x^{b_0}$, $0 \leq b_0 < 1$, and boundary conditions $y(0) = A$, $y(1) = B$ (A, B are finite constants) have been extended for general class of nonnegative functions $p(x) = x^{b_0}g(x)$, $0 \leq b_0 < 1$, and the boundary conditions $y(0) = A$, $\alpha y(1) + \beta y'(1) = \gamma$, or, $y'(0) = 0$, $\alpha y(1) + \beta y'(1) = \gamma$. The second order convergence of the methods have been established for general non-negative function $p(x)$ and under quite general conditions on $f(x, y)$. Both methods reduce to classical methods in the case $b_0 = 0$ and $g(x) = 1$ except for the method based on a uniform mesh with boundary condition $y'(0) = 0$. Numerical examples for general nonnegative function $p(x)$ illustrate the order of convergence of both methods.

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1. Introduction

Consider the class of weakly regular singular two-point boundary value problems

$$(p(x)y')' = f(x, y), \quad 0 < x \leq 1, \quad (1)$$

$$y(0) = A, \quad \alpha y(1) + \beta y'(1) = \gamma, \quad (2)$$

$$\text{or } y'(0) = 0, \quad \alpha y(1) + \beta y'(1) = \gamma, \quad (3)$$

where $\alpha > 0$, $\beta \geq 0$ and A, γ finite constants. We assume that $p(x)$ satisfies the following conditions:

- (A) (i) $p(x) > 0$ on $(0, 1]$,
- (ii) $p(x) \in C^1(0, 1]$,
- (iii) $p(x) = x^{b_0}g(x)$ on $[0, 1]$ and $b_0 \in [0, 1)$ and for some $r > 1$

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$G(x) = 1/g(x)$ is analytic in $\{z : |z| < r\}$ with Taylor's expansion

$$G(x) = G_k + (x - x_k)G'_k + (1/2!)(x - x_k)^2 G''_k + (1/3!)(x - x_k)^3 G'''(\zeta), \quad (4)$$

where ζ lies between x and x_k . Further we assume that

(B1) for $(x, y) \in \{[0, 1] \times \mathbb{R}\}$, $f(x, y)$ is continuous, $\partial f / \partial y$ exists, it is continuous and $\partial f / \partial y \geq 0$,

(B2) f , $x^{b_0} f'$ and $x^{b_0+1} f''$ are bounded on $(0, 1]$, or

(B3) f , f'' and f''' are bounded on $(0, 1]$.

Existence-uniqueness of the problem (1) with boundary conditions $y(0) = A$ (or $y'(0) = 0$) and $y(1) = B$ have been established in [7,8] with non-linear forcing term as $p(x) f(x, y)$ where the conditions on $p(x)$ satisfy (A)(i)–(iii). Existence-uniqueness for more general problem of (1) has been established in [5] with non-linear boundary condition at $x = 1$.

There is considerable literature on numerical methods for such problems (see, e.g., [1–5,9]). In [2,9] second order methods while in [1,3,4] fourth order methods are developed. Most of the authors have considered the problem (1) for the function $p(x) = x^{b_0}$, $0 \leq b_0 < 1$, with the boundary conditions $y(0) = A$ and $y(1) = B$.

In this work we describe two methods for the problem (1) with boundary conditions (2) or (3), first one is based on a non-uniform mesh and second one is based on a uniform mesh. These methods extend the methods M_1 and M_2 developed by Chawla and Katti [2] for $p(x) = x^{b_0}$ to a general class of non-negative functions $p(x)$ satisfying conditions in (A). Also, for $b_0 = 0$ and $g(x) = 1$ both methods reduce to classical second order methods based on one evaluation of $f(x, y)$ except for the second method with boundary conditions (3). In Section 3 we establish the order of accuracy of the methods for general class of functions $p(x)$ and under quite general conditions on $f(x, y)$. In the case of $p(x) = x^{b_0}$, $0 \leq b_0 < 1$, and the boundary conditions $y(0) = A$ and $y(1) = B$ our second method based on three evaluations of f provides better results than the method given in [6] which is also based on three evaluations. This is corroborated by one example in Section 4 (Table 1). To illustrate the convergence and to corroborate the order of accuracy of the methods, we applied the methods on two examples for general class of non-negative functions $p(x)$.

2. Description of the methods

We describe the methods in two parts:

- (i) First method: based on a non-uniform mesh and
- (ii) Second method: based on a uniform mesh.

2.1. First method: based on a non-uniform spacing

For a positive integer $N \geq 2$, consider a non-uniform mesh over $[0, 1]: 0 = x_0 < x_1 < x_2 < \dots < x_N = 1$. Denote $y_k = y(x_k)$, $f_k = f(x_k, y_k)$, etc. We set $z(x) = p(x)y'$ and integrating (1) from x_k to t then dividing by $p(x)$ and again integrating from x_k to x_{k+1} and changing the order of integration we get

$$y_{k+1} - y_k = z_k J_k + \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} (p(\tau))^{-1} d\tau \right) f(t) dt, \quad (5)$$

where we have set

$$J_k = \int_{x_k}^{x_{k+1}} (p(\tau))^{-1} d\tau. \quad (6)$$

In an analogous way, we obtain,

$$y_k - y_{k-1} = z_k J_{k-1} - \int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} (p(\tau))^{-1} d\tau \right) f(t) dt. \quad (7)$$

Eliminating z_k from (5) and (7) we obtain the identity

$$(y_{k+1} - y_k)/J_k - (y_k - y_{k-1})/J_{k-1} = I_k^+/J_k + I_k^-/J_{k-1}, \quad k = 1(1)(N-1), \quad (8)$$

where

$$I_k^\pm = \int_{x_k}^{x_{k\pm 1}} \left(\int_t^{x_{k\pm 1}} (p(\tau))^{-1} d\tau \right) f(t) dt. \quad (9)$$

We are interested here in a second order method based on one evaluation of f . Using the Taylor expansion given in (A)(iii) we get

$$I_k^\pm = \sum_{i=0}^1 f_k^{(i)} \left[\sum_{j=0}^2 \frac{1}{j!} A_{ij,k}^\pm G_k^{(j)} + \frac{1}{6} A_{i3,k}^\pm G_k'''(\zeta_k^\pm) \right] + \frac{1}{2} f''(\zeta_k^\pm) \left[\sum_{j=0}^2 \frac{1}{j!} A_{2j,k}^\pm G_k^{(j)} + \frac{1}{6} A_{23,k}^\pm G_k'''(\zeta_k^\pm) \right], \quad x_{k-1} < \zeta_k^-, \zeta_k^- < x_k < \zeta_k^+, \zeta_k^+ < x_{k+1}, \quad (10)$$

where

$$A_{ij,k}^\pm = \frac{1}{(i+1)!} \sum_{m=0}^{i+j+1} \frac{(-1)^m}{(i+j+2-b_0-m)!} \binom{i+j+1}{m} x_k^m (x_{k\pm 1}^{i+j+2-b_0-m} - x_k^{i+j+2-b_0-m}),$$

$$i = 0(1)2, \quad j = 0(1)3. \quad (11)$$

Now from (8) and (10) we get

$$-y_{k-1}/J_{k-1} + (1/J_k + 1/J_{k-1})y_k - y_{k+1}/J_k + B_{00,k} G_k f_k + t_k^{(1)} = 0, \quad k = 1(1)(N-1), \quad (12)$$

where

$$t_k^{(1)} = \sum_{j=0}^2 \frac{1}{j!} G_k^{(j)} \left[\frac{1}{2} B_{2j,k} f''(\zeta_k) + B_{1j,k} f'_k \right] + f_k \sum_{j=1}^2 \frac{1}{j!} B_{0j,k} G_k^{(j)} + \frac{1}{12} G_k'''(\zeta_k) \left[2 \sum_{i=0}^1 B_{i3,k} f_k^{(i)} + B_{23,k} f''(\zeta_k) \right], \quad x_{k-1} < \zeta_k, \zeta_k < x_{k+1} \quad (13)$$

and $B_{ij,k} = (A_{ij,k}^+/J_k + A_{ij,k}^-/J_{k-1})$, $i = 0(1)2$, $j = 0(1)3$. Now from Eq. (7) for $k = N$ and the boundary condition at $x = 1$ we get

$$-y_{N-1}/J_{N-1} + (1/J_{N-1} + \alpha/\beta G_N)y_N + (A_{00,N}^-/J_{N-1})G_N f_N + t_N^{(1)} = \gamma/\beta G_N, \quad (14)$$

where

$$t_N^{(1)} = \frac{1}{J_{N-1}} \left[\sum_{j=0}^2 \frac{1}{j!} G_N^{(j)} \left\{ \frac{1}{2} A_{2j,N}^- f''(\zeta_N^-) + A_{1j,N}^- f'_N \right\} + f_N \sum_{j=1}^2 \frac{1}{j!} A_{0j,N}^- G_N^{(j)} + \frac{1}{12} G_N'''(\zeta_N^-) \left\{ 2 \sum_{i=0}^1 A_{i3,N}^- f_N^{(i)} + A_{23,N}^- f''(\zeta_N^-) \right\} \right], \quad x_{N-1} < \zeta_N^-, \zeta_N^- < x_N. \quad (15)$$

In the case of boundary conditions (3) we also require the discretization for $k = 1$. For this, integrating $z' = f$ from 0 to x_1 and from Eq. (5) for $k = 1$ we obtain the identity

$$(y_2 - y_1)/J_1 = I_1 + I_1^+/J_1, \quad (16)$$

where

$$I_1 = A_{00,1}^- f_1 + A_{10,1}^- f_1' + \frac{1}{2} A_{20,1}^- f_1''(\xi_1^-), \quad 0 < \xi_1^- < x_1, \quad (17)$$

$A_{00,1}^- = x_1$, $A_{10,1}^- = -x_1^2/2$, $A_{20,1}^- = x_1^3/3$ and hence for $k = 1$, we get the discretization as follows:

$$y_1/J_1 - y_2/J_1 + B_{00,1} G_1 f_1 + t_1^{(1)} = 0, \quad (18)$$

where

$$\begin{aligned} t_1^{(1)} = & \frac{1}{2} f_1''(\xi_1) \left[B_{20,1} G_1 + \frac{1}{J_1} \sum_{j=1}^2 \frac{1}{j!} A_{2j,1}^+ G_1^{(j)} + \frac{1}{6J_1} A_{23,1}^+ G_1'''(\xi_1) \right] + B_{10,1} G_1 f_1' \\ & + \frac{1}{J_1} \sum_{i=0}^1 \left[\sum_{j=1}^2 A_{ij,1}^+ G_1^{(j)} + \frac{1}{6} A_{i3,1}^+ G_1'''(\xi_1) \right] f_1^{(i)}, \quad 0 < \xi_1, \zeta_1 < x_2 \end{aligned} \quad (19)$$

and $B_{m0,1} = (A_{m0,1}^+/J_1 + A_{m0,1}^-/G_1)$, $m = 0, 1, 2$.

To compute y_0 we use

$$y_0 = y_1 - x_1^{2-b_0} G_1 f_1/(2 - b_0) + t_0 \quad (20)$$

which is obtained by integrating $z' = f$ twice, first from 0 to x , then from 0 to x_1 and then by interchanging the order of integration, t_0 is of order $h^{3+(2-b_0)/(1-b_0)}$ for the mesh $x_k = (kh)^{1/(1-b_0)}$.

2.2. Second method: based on a uniform spacing

In this section we describe a method based on a uniform mesh spacing. For this Eqs. (12)–(13) can be modified as

$$\begin{aligned} -y_{k-1}/J_{k-1} + (1/J_k + 1/J_{k-1})y_k - y_{k+1}/J_k + (B_{00,k} G_k + B_{01,k} G_k') f_k \\ + B_{10,k} G_k (f_{k+1} - f_{k-1})/2h + t_k^{(2)} = 0, \quad k = 1(1)(N-1), \end{aligned} \quad (21)$$

where

$$\begin{aligned} t_k^{(2)} = & -\frac{h^2}{6} B_{10,k} G_k f_k'''(\sigma_k) + \frac{1}{2} f_k''(\xi_k) \sum_{j=0}^2 \frac{1}{j!} B_{2j,k} G_k^{(j)} + B_{11,k} G_k' f_k' \\ & + \frac{1}{2} \sum_{i=0}^1 \left[B_{i2,k} G_k'' + \frac{1}{3} B_{i3,k} G_k'''(\xi_k) \right] f_k^{(i)} + \frac{1}{12} B_{23,k} G_k'''(\xi_k) f_k''(\xi_k) \end{aligned} \quad (22)$$

by taking central difference approximation for f_k' as $f_k' = (f_{k+1} - f_{k-1})/2h - (h^2/6) f_k'''(\sigma_k)$, $\sigma_k \in (x_{k-1}, x_{k+1})$.

Similarly, (18)–(19) can be modified as

$$\begin{aligned} y_1/J_1 - y_2/J_1 + [(B_{00,1} - B_{10,1})G_1 + A_{01,1}^+ G_1'/J_1] f_1 \\ + (1/h) B_{10,1} G_1 f_2 + t_1^{(2)} = 0, \end{aligned} \quad (23)$$

where

$$\begin{aligned} t_1^{(2)} = & (1/2)[B_{20,1}f''(\xi_1) - hB_{10,1}f''(\sigma_1)]G_1 + (1/2J_1)[2A_{11,1}^+G_1'f_1' \\ & + f''(\xi_1)\sum_{j=1}^2 \frac{1}{j!}A_{2j,1}^+G_1^{(j)} + \sum_{i=0}^1 f_1^{(i)} \left\{ A_{i2,1}^+G_1'' + \frac{1}{3}A_{i3,1}^+G_1'''(\xi_1) \right\} \\ & + (1/6)A_{23,1}^+G_1'''(\xi_1)f''(\xi_1)] \end{aligned} \quad (24)$$

by using $f_1' = (f_2 - f_1)/h - (h/2)f''(\sigma_1)$, $x_1 < \sigma_1 < x_2$. For boundary condition at $x = 1$ we use (14) and to compute y_0 Eq. (20) can be used in which t_0 is of order h^{3-b_0} for a uniform mesh.

3. Convergence of the methods

In this section we establish the convergence of both methods.

3.1. Convergence of the first method

3.1.1. Case I: for boundary conditions $y(0) = A$, $\alpha y(1) + \beta y'(1) = \gamma$

Let $F(Y) = (f_1, \dots, f_N)^T$, $Y = (y_1, \dots, y_N)^T$, $T = (t_1^{(1)}, \dots, t_N^{(1)})^T$, and $Q = (q_1, 0, \dots, 0, q_N)^T$, the discretizations (12) and (14) for the solution of (1)–(2) can be expressed in matrix form as

$$DY + PF(Y) + T = Q, \quad (25)$$

where $D = (d_{ij})$ and $P = (p_{ij})$ are $(N \times N)$ tridiagonal matrix, and diagonal matrix, respectively, with $d_{k,k-1} = -1/J_{k-1}$, $k = 2(1)N$, $d_{k,k+1} = -1/J_k$, $k = 1(1)(N-1)$, $d_{k,k} = (1/J_k + 1/J_{k-1})$, $k = 1(1)(N-1)$, $d_{N,N} = (1/J_{N-1} + \alpha/\beta G_N)$, $p_{k,k} = B_{00,k}G_k$, $k = 1(1)(N-1)$, $p_{N,N} = A_{00,N}^-G_N/J_{N-1}$ and $q_1 = A/J_0$, $q_N = \gamma/\beta G_N$.

Thus, the method to approximate the solution Y can be written as

$$D\tilde{Y} + PF(\tilde{Y}) = Q, \quad (26)$$

where \tilde{Y} is approximation to Y . Now from (25) and (26) we get the error equation

$$(D + PM)E = T, \quad (27)$$

where $E = \tilde{Y} - Y$, $F(\tilde{Y}) - F(Y) = ME$, $M = \text{diag}\{U_1, \dots, U_N\}$ (where $U_k = \partial f_k / \partial y_k \geq 0$). Take $x_k = (kh)^{1/(1-b_0)}$ and for fixed x_k and $h \rightarrow 0$, it can be shown from (11) and (6) that

$$A_{ij,k}^+ = \frac{(i+j+1)h^{i+j+2}x_k^{(i+j+1)b_0}}{(i+1)(1-b_0)^{i+j+2}} \left[\frac{1}{(i+j+1)(i+j+2)} + \frac{b_0hx_k^{b_0-1}}{2(i+j+3)(1-b_0)} + O(h^2) \right], \quad (28)$$

$$A_{ij,k}^- = \frac{(i+j+1)h^{i+j+2}x_k^{(i+j+1)b_0}}{(i+1)(1-b_0)^{i+j+2}} \left[\frac{(-1)^{i+j}}{(i+j+1)(i+j+2)} + \frac{(-1)^{i+j+1}b_0hx_k^{b_0-1}}{2(i+j+3)(1-b_0)} + O(h^2) \right],$$

$$J_k = \frac{hG_k}{(1-b_0)} \left[1 + \frac{hx_k^{b_0}}{2(1-b_0)} \frac{G_k'}{G_k} + O(h^2) \right],$$

$$J_{k-1} = \frac{hG_k}{(1-b_0)} \left[1 - \frac{hx_k^{b_0}}{2(1-b_0)} \frac{G_k'}{G_k} + O(h^2) \right], \quad (29)$$

and hence

$$B_{ij,k} = \begin{cases} \frac{2h^{i+j+1}x_k^{(i+j+1)b_0}}{(i+1)(i+j+2)(1-b_0)^{i+j+1}G_k} + O(h^{i+j+3}), & i+j=0, 2, 4, \\ \frac{h^{i+j+2}x_k^{(i+j+2)b_0-1}}{(i+1)(1-b_0)^{i+j+2}G_k} \left[\frac{(i+j+1)b_0}{(i+j+3)} - \frac{x_k}{(i+j+2)} \frac{G'_k}{G_k} \right] & i+j=1, 3, 5. \\ +O(h^{i+j+4}), \end{cases} \quad (30)$$

With the help of Eqs. (28)–(30), it is easy to see that D and $D + PM$ are irreducible, monotone and in view of $PM \geq 0$ we get D^{-1} , $(D + PM)^{-1}$ exists, is non-negative and $(D + PM)^{-1} \leq D^{-1}$.

Let $Z = (1, \dots, 1)^T$ and $S^* = (S_1^*, \dots, S_N^*)^T = DZ$ denote the vector of row-sums of D . Also, let $V = (V_1, \dots, V_N)^T$ where $V_j = (2\beta/\alpha) + 2 - (x_j + 1)^2/2$ and let $R = (R_1, \dots, R_N)^T = DV$. We next obtain bound for $D^{-1} = (d_{ij}^{-1})$.

Since

$$R_k = \frac{1}{2J_{k-1}}[x_{k-1}^2 + 2x_{k-1} - x_k^2 - 2x_k] + \frac{1}{2J_k}[x_{k+1}^2 + 2x_{k+1} - x_k^2 - 2x_k]$$

which gives

$$\begin{aligned} R_k = & \frac{1}{J_{k-1}} \left[\frac{h}{(1-b_0)} x_k^{b_0+1} \left\{ -1 + \frac{h(b_0+1)}{(1-b_0)} x_k^{b_0-1} + O(h^2) \right\} \right. \\ & + \frac{h}{(1-b_0)} x_k^{b_0} \left\{ -1 + \frac{b_0 h}{2(1-b_0)} x_k^{b_0-1} + O(h^2) \right\} \Big] \\ & + \frac{1}{J_k} \left[\frac{h}{(1-b_0)} x_k^{b_0+1} \left\{ 1 + \frac{h(b_0+1)}{(1-b_0)} x_k^{b_0-1} + O(h^2) \right\} \right. \\ & + \frac{h}{(1-b_0)} x_k^{b_0} \left\{ 1 + \frac{b_0 h}{2(1-b_0)} x_k^{b_0-1} + O(h^2) \right\} \Big], \end{aligned}$$

and hence for sufficiently small h ,

$$R_k > \{b_0 h / (1-b_0) G_k\} x_k^{2b_0-1}, \quad k = 2(1)(N-1). \quad (31)$$

In the same way

$$\begin{aligned} R_1 = & \frac{1}{J_0} \left[\frac{2\beta}{\alpha} + \frac{3}{2} - \frac{1}{2} h^{2/(1-b_0)} - h^{1/(1-b_0)} \right] \\ & + \frac{1}{2J_1} h^{1/(1-b_0)} (2^{1/(1-b_0)} - 1) [h^{1/(1-b_0)} (2^{1/(1-b_0)} + 1) + 2], \\ R_N = & \frac{2h}{(1-b_0)J_{N-1}} \left[-1 + \frac{(2b_0+1)h}{4(1-b_0)} + O(h^2) \right] + \frac{2}{G_N} \end{aligned}$$

which gives $R_1, R_N > 0$ for small h . Now from $D^{-1}R = V$ we get

$$\{b_0 h / (1-b_0)\} \sum_{k=2}^{N-1} (d_{i,k}^{-1} / G_k) x_k^{2b_0-1} \leq V_i < (2\beta/\alpha) + 3/2, \quad i = 1(1)N. \quad (32)$$

Use Eqs. (28)–(30) and let there exist constants N_i , $i = 0(1)2$, such that

$$x^{b_0+1} |f''| \leq N_2, \quad x^{b_0} |f'| \leq N_1, \quad |f| \leq N_0 \quad \forall 0 < x \leq 1, \quad (33)$$

then from (13) and (15) we get

$$|t_k^{(1)}| \leq (h^3/2(1-b_0)^3) x_k^{2b_0-1} N_3, \quad k = 1(1)(N-1), \quad (34)$$

and

$$|t_N^{(1)}| \leq (h^2/4(1-b_0)^3)N'_3 \quad (35)$$

for sufficiently small h where N_3, N'_3 are suitable chosen constants.

Now since $S_1^* = 1/J_0$ and $S_N^* = \alpha/\beta G_N$, with the help of $D^{-1}S^* = Z$ we obtain

$$d_{i,1}^{-1} \leq 1/S_1^* = J_0 \Rightarrow d_{i,1}^{-1} \leq hG_0/(1-b_0), \quad d_{i,N}^{-1} \leq (\beta/\alpha)G_N, \quad i = 1(1)N, \quad (36)$$

and in view of $(D + PM)^{-1} \leq D^{-1}$ we have $\|E\|_\infty \leq \|D^{-1}T\|_\infty$ and thus from (27), (32), (34)–(36) it is easy to establish that

$$\|E\|_\infty = O(h^2).$$

3.1.2. Case II: for boundary conditions $y'(0) = 0, \alpha y(1) + \beta y'(1) = \gamma$

The discretization (12), (14), (18) for the solution of the singular boundary value problem (1) with boundary conditions (3) can be expressed in matrix form by (25) with $D = (d_{ij})$, a tridiagonal matrix, and $P = (p_{ij})$, a diagonal matrix, where $d_{k,k-1} = -1/J_{k-1}, k = 2(1)N, d_{k,k+1} = -1/J_k, k = 1(1)(N-1), d_{1,1} = 1/J_1, d_{k,k} = (1/J_k + 1/J_{k-1}), k = 2(1)(N-1), d_{N,N} = (1/J_{N-1} + \alpha/\beta G_N), p_{k,k} = B_{00,k}G_k, k = 1(1)(N-1), p_{N,N} = A_{00,N}^- G_N/J_{N-1}$ and $Q = (0, 0, \dots, 0, q_N)^T$ with $q_N = \gamma/\beta G_N$.

Now following the convergence analysis of Case I and using $R_1 > (1-b_0)x_1^{b_0}/G_1, d_{i,1}^{-1} < G_1(2\beta/\alpha + \frac{3}{2})/(1-b_0)x_1^{b_0}$ and $|t_1^{(1)}| \leq h^2 x_1^{b_0} N_3''/(1-b_0)^3$ for suitable constant N_3'' , it is easy to establish that

$$\|E\|_\infty = O(h^2).$$

Thus we have established the following result:

Theorem 1. Assume that $f(x, y)$ satisfies (B1)–(B2) and $p(x)$ satisfies conditions in (A). Then, the first method based on (12) and (14) (in case of boundary value problem (1)–(2)) or (12), (14) and (18) (in case of boundary value problem (1), and (3)) for a non-uniform mesh with $x_k = (kh)^{1/(1-b_0)}$ is of second order accuracy for sufficiently small h .

3.2. Convergence of the second method

3.2.1. Case I: for boundary conditions $y(0) = A, \alpha y(1) + \beta y'(1) = \gamma$

The discretization (14), (21) for the solution of the singular boundary value problem (1)–(2) can be expressed in matrix form by (25) where $D = (d_{ij})$ and $P = (p_{ij})$ are tridiagonal matrices with $d_{k,k-1} = -1/J_{k-1}, k = 2(1)N, d_{k,k+1} = -1/J_k, k = 1(1)(N-1), d_{k,k} = (1/J_k + 1/J_{k-1}), k = 1(1)(N-1), d_{N,N} = (1/J_{N-1} + \alpha/\beta G_N), p_{k,k-1} = -(1/2h)B_{10,k}G_k, k = 2(1)(N-1), p_{N,N-1} = 0, p_{k,k+1} = (1/2h)B_{10,k}G_k, k = 1(1)(N-1), p_{N,N} = A_{00,N}^- G_N/J_{N-1}, p_{k,k} = (B_{00,k}G_k + B_{01,k}G'_k), k = 1(1)(N-1)$ and $q_1 = A/J_0 + (1/2h)B_{10,1}G_1 f_0, q_N = \gamma/\beta G_N$.

Thus, the method to approximate the solution Y can be written as (26) and hence we get the error equation (27).

Let $h_0 = \min\{h_1, h_2\}$ where $h_1 = \{8/[U^* b_0 \sup_{[0,1]} |G(x)|]\}^{1/(2-b_0)}, h_2 = \{1/[2b_0 \sup_{[0,1]} |G'(x)/G(x)|]\}$ and $U^* = \sup_{[0,1] \times \mathbf{R}} (\partial f / \partial y)$, then for fixed x_k and $h \rightarrow 0$ ($h < h_0$), it can be shown that

$$\begin{aligned} A_{ij,k}^+ &= \frac{h^{i+j+2}}{(i+1)} x_k^{-b_0} \left[\frac{1}{(i+j+2)} - \frac{b_0 h}{(i+j+3)x_k} + O(h^2) \right], \\ A_{ij,k}^- &= \frac{(-1)^{i+j} h^{i+j+2}}{(i+1)} x_k^{-b_0} \left[\frac{1}{(i+j+2)} + \frac{b_0 h}{(i+j+3)x_k} + O(h^2) \right], \end{aligned} \quad (37)$$

$$\begin{aligned} J_k &= hG_k x_k^{-b_0} \left[1 - \frac{h}{2} \left\{ \frac{b_0}{x_k} - \frac{G'_k}{G_k} \right\} + O(h^2) \right], \\ J_{k-1} &= hG_k x_k^{-b_0} \left[1 + \frac{h}{2} \left\{ \frac{b_0}{x_k} - \frac{G'_k}{G_k} \right\} + O(h^2) \right], \end{aligned} \quad (38)$$

and hence

$$B_{ij,k} = \begin{cases} \frac{2h^{i+j+1}}{(i+1)(i+j+2)G_k} + O(h^{i+j+3}), & i+j=0, 2, 4, \\ -\frac{h^{i+j+2}}{(i+1)(i+j+2)x_k G_k} \left[\frac{(i+j+1)b_0}{(i+j+3)} + \frac{x_k G'_k}{G_k} \right] & \\ + O(h^{i+j+4}), & i+j=1, 3, 5. \end{cases} \quad (39)$$

Let $Z = (1, \dots, 1)^T$ and $S = (S_1, \dots, S_N)^T = (D + PM)Z$ denote the vector of row-sums of $D + PM$, then for sufficiently small h ,

$$S_1 > 1/2J_0, \quad S_N > \alpha/\beta G_N, \quad S_k > hU_k/4, \quad k = 2(1)(N-1). \quad (40)$$

In view of (40) and $h < h_0$ it is easy to see that $D + PM$ is irreducible and monotone. We now assume that $\partial f / \partial y > 0$. Let $U_* = \min \partial f / \partial y$, then $U_* > 0$ and we get

$$S_k > hU_*/4, \quad k = 2(1)(N-1). \quad (41)$$

Since $(D + PM)^{-1}S = Z$ with the help of (40) and (41) it follows for sufficiently small h :

$$(D + PM)_{i,1}^{-1} \leq 1/S_1 < 2J_0 < 4h^{1-b_0}G_0/(1-b_0), \quad (D + PM)_{i,N}^{-1} \leq \beta G_N/\alpha, \quad i = 1(1)N \quad (42)$$

and

$$\sum_{k=2}^{N-1} (D + PM)_{i,k}^{-1} \leq 1/\min_{2 \leq k \leq N-1} S_k \leq 4/hU_*, \quad i = 1(1)N. \quad (43)$$

Let there exist constants \bar{N}_i , $i = 0(1)3$ such that $|f^{(i)}| \leq \bar{N}_i$, for $i = 0(1)3$, and $0 < x \leq 1$. Now use Eqs. (37)–(39), then from (22) and (15) we get

$$|t_k^{(2)}| \leq Ch^3, \quad k = 1(1)(N-1) \quad (44)$$

and

$$|t_N^{(2)}| \leq \bar{C}h^2 \quad (45)$$

for sufficiently small h where C, \bar{C} are constants, and thus from Eqs. (27), (42)–(45) we get

$$|e_i| \leq C^*h^2, \quad (46)$$

where $C^* = \{4C/(1-b_0)\} \sup_{[0,1]} |G(x)| + 4C/U_* + (\bar{C}\beta/\alpha) \sup_{[0,1]} |G(x)|$ and hence

$$\|E\|_\infty = O(h^2).$$

3.2.2. Case II: for boundary conditions $y'(0) = 0$, $\alpha y(1) + \beta y'(1) = \gamma$

The discretization (14), (21), (23) for the solution of the singular boundary value problem (1) with boundary conditions (3) can be expressed in matrix form by (25) where $D = (d_{ij})$ and $P = (p_{ij})$ are tridiagonal matrices with $d_{k,k-1} = -1/J_{k-1}$, $k = 2(1)N$, $d_{k,k+1} = -1/J_k$, $k = 1(1)(N-1)$, $d_{1,1} = 1/J_1$, $d_{k,k} = (1/J_k + 1/J_{k-1})$, $k = 2(1)(N-1)$, $d_{N,N} = (1/J_{N-1} + \alpha/\beta G_N)$, $p_{k,k-1} = -(1/2h)B_{10,k}G_k$, $k = 2(1)(N-1)$, $p_{N,N-1} = 0$, $p_{k,k+1} = (1/2h)B_{10,k}G_k$, $k = 2(1)(N-1)$, $p_{1,2} = (1/h)B_{10,1}G_1$, $p_{1,1} = (B_{00,1} - B_{10,1})G_1 + A_{01,1}^+ G'_1/J_1$, $p_{k,k} = (B_{00,k}G_k + B_{01,k}G'_k)$, $k = 2(1)(N-1)$, $p_{N,N} = A_{00,N}^- G_N/J_{N-1}$ and $Q = (0, 0, \dots, 0, q_N)^T$ with $q_N = \gamma/\beta G_N$.

Now following the convergence analysis of Case-I and using $(D + PM)_{i,1}^{-1} < 4/hU_*$, $i = 1(1)N$, and $|t_1^{(2)}| \leq \hat{C}h^3$ for suitable constant \hat{C} , it is easy to establish that

$$\|E\|_\infty = O(h^2).$$

Thus we have established the following result:

Theorem 2. Assume that $f(x, y)$ satisfies (B1), (B3) and $p(x)$ satisfies conditions in (A). Then, the second method based on (21) and (14) (in case of boundary conditions (2)) or (21), (14) and (23) (in case of boundary conditions (3)) for a uniform mesh is of second order accuracy for sufficiently small h .

Remark 1. For $b_0 = 0$ and $g(x) = 1$, $B_{10,k} = 0$ and $B_{01,k} = 0$, hence the first method for both type of boundary conditions (2) or (3) and second method for boundary conditions (2) reduce to classical method for $y'' = f(x, y)$ based on one evaluation of f .

4. Numerical illustrations

In this section, we illustrate the methods and corroborate the order of convergence of the methods. We also compare our second method with the method given in [6] as both are based on three evaluations of f .

We have compared the method given in [6] with the second method of this work for Example 1. The maximum absolute errors are displayed in Table 1 which shows that our method is superior to that of [6]. To illustrate the convergence and to corroborate the order of accuracy of the methods given in this work for general non-negative functions $p(x)$, we apply the methods to two examples, Examples 2 and 3. Tables 2 and 3, respectively, display the results of maximum absolute errors and order of convergence (accuracy) for Examples 2 and 3 for $b_0 = 0.60$ with $h = 2^k$, $k = 4(1)9$, which shows that the methods work well and are of second order accuracy.

Example 1.

$$(x^{b_0} y')' = \lambda x^{b_0 + \lambda - 2} ((b_0 + \lambda - 1) + \lambda x^\lambda) y,$$

$$y(0) = 1 \quad y(1) = e$$

with exact solution $y(x) = \exp(x^\lambda)$ and $\lambda \geq 2$.

Table 1
Maximum absolute error for Example 1

N	$b_0 = 0.5, \lambda = 4.0$		$b_0 = 0.5, \lambda = 3.75$	
	Our method	Method in [6]	Our method	Method in [6]
16	1.15(−2) ^a	2.1(−2)	1.14(−2)	6.8(−2)
32	2.90(−3)	5.2(−3)	2.86(−3)	1.7(−2)
64	7.28(−4)	1.3(−3)	7.15(−4)	4.2(−3)
128	1.82(−4)	3.3(−4)	1.79(−4)	1.0(−3)

^a1.15(−2) = 1.15×10^{-2} .

Table 2
Maximum absolute error for Example 2

N/ b_0	First method				Second method			
	$y(0) = -\ln 4$		$y'(0) = 0$		$y(0) = -\ln 4$		$y'(0) = 0$	
	0.60	Order	0.60	Order	0.60	Order	0.60	Order
16	1.15(−2)		1.34(−2)		5.58(−3)		8.36(−3)	
32	2.81(−3)	2.03	3.26(−3)	2.04	1.42(−3)	1.98	2.09(−3)	2.00
64	6.97(−4)	2.01	8.08(−4)	2.01	3.56(−4)	1.99	5.24(−4)	1.99
128	1.74(−4)	2.00	2.01(−4)	2.00	8.93(−5)	2.00	1.32(−4)	2.00
256	4.35(−5)	2.00	5.03(−5)	2.00	2.24(−5)	2.00	3.29(−5)	2.00
512	1.09(−5)	2.00	1.26(−5)	2.00	5.60(−6)	2.00	8.24(−6)	2.00

Table 3
Maximum absolute error for Example 3

N/b_0	First method				Second method			
	$y(0) = 1$		$y'(0) = 0$		$y(0) = 1$		$y'(0) = 0$	
	0.60	Order	0.60	Order	0.60	Order	0.60	Order
16	3.95(−1)		3.96(−1)		9.38(−2)		9.40(−2)	
32	1.26(−1)	1.65	1.26(−1)	1.65	2.48(−2)	1.92	2.49(−2)	1.92
64	3.39(−2)	1.89	3.40(−2)	1.89	6.32(−3)	1.97	6.33(−3)	1.97
128	8.66(−3)	1.97	8.68(−3)	1.97	1.59(−3)	1.99	1.59(−3)	1.99
256	2.18(−3)	1.99	2.18(−3)	1.99	3.98(−4)	2.00	3.99(−4)	2.00
512	5.45(−4)	2.00	5.46(−4)	2.00	9.97(−5)	2.00	1.00(−5)	2.00

Example 2.

$$(x^{b_0} e^x y')' = 5e^x x^{b_0+3} (5x^5 e^y - (b_0 + 4) - x)/(4 + x^5),$$

$$y(0) = \ln(1/4) \text{ (or } y'(0) = 0), \quad y(1) + 5y'(1) = \ln(1/5) - 5$$

with exact solution $y(x) = -\ln(4 + x^5)$.

Example 3.

$$(x^{b_0} (1 + x^2) y')' = 5(1 + x^2) x^{b_0+3} (5x^5 + (b_0 + 4) + 2x^2/(1 + x^2))y,$$

$$y(0) = 1 \text{ (or } y'(0) = 0), \quad y(1) + y'(1) = 6e$$

with exact solution $y(x) = \exp(x^5)$.

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